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**Superintegrability of the Tremblay–Turbiner–
Winternitz quantum Hamiltonians on a plane
for odd k**

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Online at stacks.iop.org/JPhysA/43/082001**Abstract**

In a recent communication paper by Tremblay *et al* (2009 *J. Phys. A: Math. Theor.* **42** 205206), it has been conjectured that for any integer value of k , some novel exactly solvable and integrable quantum Hamiltonian H_k on a plane is superintegrable and that the additional integral of motion is a $2k$ th-order differential operator Y_{2k} . Here we demonstrate the conjecture for the infinite family of Hamiltonians H_k with odd $k \geq 3$, whose first member corresponds to the three-body Calogero–Marchioro–Wolfes model after elimination of the centre-of-mass motion. Our approach is based on the construction of some D_{2k} -extended and invariant Hamiltonian \mathcal{H}_k , which can be interpreted as a modified boson oscillator Hamiltonian. The latter is then shown to possess a D_{2k} -invariant integral of motion \mathcal{Y}_{2k} , from which Y_{2k} can be obtained by projection in the D_{2k} identity representation space.

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1. Introduction

In a recent work, Tremblay, Turbiner and Winternitz (TTW) have introduced an infinite family of exactly solvable quantum Hamiltonians on a plane:

$$H_k = -\partial_r^2 - \frac{1}{r}\partial_r - \frac{1}{r^2}\partial_\varphi^2 + \omega^2 r^2 + \frac{k^2}{r^2}(\alpha \sec^2 k\varphi + \beta \csc^2 k\varphi), \quad (1.1)$$

defined in terms of polar coordinates (r, φ) and of an arbitrary real number k [1]. In (1.1), the configuration space is given by the sector $0 \leq r < \infty$, $0 \leq \varphi \leq \pi/(2k)$, and ω, α, β are three parameters such that $\omega > 0$, $\alpha = a(a-1) > -1/(4k^2)$, $\beta = b(b-1) > -1/(4k^2)$. The Hamiltonian H_k has been shown to be integrable with a second-order integral of motion

$$X_k = -\partial_\varphi^2 + k^2(\alpha \sec^2 k\varphi + \beta \csc^2 k\varphi), \quad (1.2)$$

responsible for the separation of variables of the corresponding Schrödinger equation in polar coordinates.

In the same work, it has also been conjectured that for all integer values of k , the Hamiltonian (1.1) is superintegrable and that the additional functionally-independent integral of motion is a $2k$ th-order differential operator Y_{2k} . This conjecture has been proved by a direct calculation of Y_{2k} for $k = 1, 2, 3$ and 4 , as well as for $k = 1-6$ and 8 in the special case where either α or β vanishes. Actually, for $k = 1, 2$ and 3 , the superintegrability of H_k was already known since the Hamiltonian then reduces to that of the Smorodinsky–Winternitz system ($k = 1$) [2, 3], of the rational BC_2 model ($k = 2$) [4] or of the Calogero–Marchioro–Wolfes (CMW) model ($k = 3$) [5, 6].

More recently, the credibility of the conjecture has been reinforced by showing that for the corresponding classical systems all bounded trajectories are closed and that the motion is periodic for all integer and rational values of k [7]. Such classical systems have been proved to be superintegrable [8] and generalizable to higher dimensions [9].

Since it is clear that the proof of the conjecture for $k > 4$ requires a different approach from that based on a brute-force calculation of Y_{2k} , it is worth investigating the use of some differential–difference operators or covariant derivatives, known in the mathematical literature as Dunkl operators [10]. Such a formalism, based on the introduction of exchange operators, has indeed been successful in proving the superintegrability of the N -body Calogero model [11, 12], which for $N = 3$ is related to H_3 with $\beta = 0$, as well as that of the CMW model [13], for which some results in polar coordinates are also available [14].

In a first step along these lines, we have generalized the exchange operators known for the CMW model to the whole set of Hamiltonians H_k for which k is the integer [15]. For this purpose, we have realized the elements of the dihedral group D_{2k} as operators on the plane (r, φ) and employed them to define some differential–difference operators D_r and D_φ , where the latter assumes a different form according to whether k is odd or even. We have then constructed D_{2k} -extended and invariant Hamiltonians \mathcal{H}_k , from which the starting Hamiltonians H_k can be retrieved by projection in the D_{2k} identity representation space. Furthermore, we have shown that X_k can be obtained by the same procedure from the operator $-D_\varphi^2$, commuting with \mathcal{H}_k .

The purpose of the present communication is to build on these advances in order to provide a rigorous proof of the above-mentioned conjecture for any odd k value. More specifically, we plan to demonstrate the existence of a second integral of motion Y_{2k} of H_k , obtainable by projection from a corresponding operator \mathcal{Y}_{2k} for the extended Hamiltonian \mathcal{H}_k , whenever k is odd and greater than or equal to 3.

2. Exchange operator formalism for H_k with odd k

In [15], it has been shown that for any integer k value, the $4k$ elements of the dihedral group D_{2k} can be realized as \mathcal{R}^i and $\mathcal{R}^i\mathcal{I}$, $i = 0, 1, \dots, 2k - 1$, where $\mathcal{R} = \exp\left(\frac{1}{k}\pi\partial_\varphi\right)$ is the rotation operator through angle π/k in the plane (r, φ) while $\mathcal{I} = \exp(i\pi\varphi\partial_\varphi)$ ¹ changes φ into $-\varphi$. Such operators indeed satisfy the defining relations of (a unitary representation of) D_{2k} ,

$$\mathcal{R}^{2k} = \mathcal{I}^2 = 1, \quad \mathcal{I}\mathcal{R} = \mathcal{R}^{2k-1}\mathcal{I}, \quad \mathcal{R}^\dagger = \mathcal{R}^{2k-1}, \quad \mathcal{I}^\dagger = \mathcal{I}. \quad (2.1)$$

¹ In the definition of \mathcal{I} and only there, i stands for the square root of -1 .

The partial derivatives ∂_r and ∂_φ can then be extended into differential–difference operators D_r and D_φ , defined for any odd k by

$$\begin{aligned} D_r &= \partial_r - \frac{1}{r}(a\mathcal{R} + b) \left(\sum_{i=0}^{k-1} \mathcal{R}^{2i} \right) \mathcal{I}, \\ D_\varphi &= \partial_\varphi + a \sum_{i=0}^{k-1} \tan \left(\varphi + \frac{i\pi}{k} \right) \mathcal{R}^{k+2i} \mathcal{I} - b \sum_{i=0}^{k-1} \cot \left(\varphi + \frac{i\pi}{k} \right) \mathcal{R}^{2i} \mathcal{I}, \end{aligned} \quad (2.2)$$

where a and b are related to the potential strengths α and β in H_k . From (1.1), it follows that D_r and D_φ fulfil the following Hermiticity, exchange and commutation relations:

$$D_r^\dagger = -D_r - \frac{1}{r} \left[1 + 2(a\mathcal{R} + b) \left(\sum_{i=0}^{k-1} \mathcal{R}^{2i} \right) \mathcal{I} \right], \quad D_\varphi^\dagger = -D_\varphi, \quad (2.3)$$

$$\mathcal{R}D_r = D_r\mathcal{R}, \quad \mathcal{I}D_r = D_r\mathcal{I}, \quad \mathcal{R}D_\varphi = D_\varphi\mathcal{R}, \quad \mathcal{I}D_\varphi = -D_\varphi\mathcal{I},$$

$$[D_r, D_\varphi] = -\frac{2}{r}(a\mathcal{R} + b) \left(\sum_{i=0}^{k-1} \mathcal{R}^{2i} \right) \mathcal{I}D_\varphi, \quad (2.4)$$

respectively. It is worth noting that the first relation in (2.3) as well as equation (2.4) significantly differ from the corresponding results for ∂_r and ∂_φ , namely $\partial_r^\dagger = -\partial_r - \frac{1}{r}$ and $[\partial_r, \partial_\varphi] = 0$.

From D_r and D_φ , one can build a D_{2k} -extended Hamiltonian

$$\mathcal{H}_k = -D_r^2 - \frac{1}{r} \left[1 + 2(a\mathcal{R} + b) \left(\sum_{i=0}^{k-1} \mathcal{R}^{2i} \right) \mathcal{I} \right] D_r - \frac{1}{r^2} D_\varphi^2 + \omega^2 r^2, \quad (2.5)$$

which can also be rewritten as

$$\mathcal{H}_k = -\partial_r^2 - \frac{1}{r} \partial_r - \frac{1}{r^2} \left[D_\varphi^2 - k(a^2 + b^2 + 2ab\mathcal{R}) \sum_{i=0}^{k-1} \mathcal{R}^{2i} \right] + \omega^2 r^2$$

with

$$\begin{aligned} D_\varphi^2 &= \partial_\varphi^2 - \sum_{i=0}^{k-1} \sec^2 \left(\varphi + \frac{i\pi}{k} \right) a(a - \mathcal{R}^{k+2i} \mathcal{I}) - \sum_{i=0}^{k-1} \csc^2 \left(\varphi + \frac{i\pi}{k} \right) b(b - \mathcal{R}^{2i} \mathcal{I}) \\ &\quad + k(a^2 + b^2 + 2ab\mathcal{R}) \sum_{i=0}^{k-1} \mathcal{R}^{2i}. \end{aligned}$$

Such a \mathcal{H}_k is clearly invariant under D_{2k} and, due to some simple trigonometric identities, gives rise to H_k after projection in the D_{2k} identity representation space, i.e. after replacing \mathcal{R} and \mathcal{I} by 1.

Furthermore, $\mathcal{X}_k = -D_\varphi^2$ is a D_{2k} invariant, commuting with \mathcal{H}_k , and whose projection in the same space is given by $X_k - k^2(a + b)^2$, with X_k defined in equation (1.2).

3. Modified boson operators for odd $k \geq 3$

Furthermore, it is worth observing that, on one hand, the starting Hamiltonian H_k has an oscillator spectrum [1] and, on the other hand, the extended Hamiltonian \mathcal{H}_k , defined in (2.5), has (apart from some additional exchange operators) a form reminiscent of that of an oscillator

Hamiltonian in polar coordinates, $h = -\partial_r^2 - \frac{1}{r}\partial_r - \frac{1}{r^2}\partial_\varphi^2 + \omega^2 r^2$. Since the latter can also be written as $h = 2\omega(h_a + h_b)$, $h_a = \frac{1}{2}\{a^\dagger, a\}$, $h_b = \frac{1}{2}\{b^\dagger, b\}$, in terms of two sets of boson creation and annihilation operators a^\dagger, a and b^\dagger, b , it seems appropriate to introduce for \mathcal{H}_k two sets of modified boson operators

$$\begin{aligned} A &= \frac{1}{\sqrt{2\omega}} \left[\cos \varphi (\omega r + D_r) - \frac{1}{r} \sin \varphi D_\varphi \right], \\ A^\dagger &= \frac{1}{\sqrt{2\omega}} \left[\cos \varphi (\omega r - D_r) + \frac{1}{r} \sin \varphi D_\varphi \right] \end{aligned} \quad (3.1)$$

and

$$\begin{aligned} B &= \frac{1}{\sqrt{2\omega}} \left[\sin \varphi (\omega r + D_r) + \frac{1}{r} \cos \varphi D_\varphi \right], \\ B^\dagger &= \frac{1}{\sqrt{2\omega}} \left[\sin \varphi (\omega r - D_r) - \frac{1}{r} \cos \varphi D_\varphi \right], \end{aligned} \quad (3.2)$$

by simply substituting D_r and D_φ for ∂_r and ∂_φ in the definition of $a, a^\dagger, b, b^\dagger$ in polar coordinates.

At this stage, it should be noted that the fact that A^\dagger and B^\dagger are the Hermitian conjugates of A and B , respectively, is not so obvious as for a^\dagger and b^\dagger with respect to a and b . To prove such a property, one has actually to use equations (2.1), (2.3) and the definition of \mathcal{R} and \mathcal{I} , as well as the commutation relations

$$[D_r, r] = 1, \quad [D_\varphi, r] = 0, \quad (3.3)$$

$$\begin{aligned} [D_r, \cos \varphi] &= \frac{1}{r} \left[2a \sum_{i=0}^{k-1} \cos \left(\varphi + \frac{i\pi}{k} \right) \cos \frac{i\pi}{k} \mathcal{R}^{k+2i} \mathcal{I} \right. \\ &\quad \left. + 2b \sum_{i=0}^{k-1} \sin \left(\varphi + \frac{i\pi}{k} \right) \sin \frac{i\pi}{k} \mathcal{R}^{2i} \mathcal{I} \right], \end{aligned} \quad (3.4)$$

$$\begin{aligned} [D_r, \sin \varphi] &= \frac{1}{r} \left[-2a \sum_{i=0}^{k-1} \cos \left(\varphi + \frac{i\pi}{k} \right) \sin \frac{i\pi}{k} \mathcal{R}^{k+2i} \mathcal{I} \right. \\ &\quad \left. + 2b \sum_{i=0}^{k-1} \sin \left(\varphi + \frac{i\pi}{k} \right) \cos \frac{i\pi}{k} \mathcal{R}^{2i} \mathcal{I} \right], \end{aligned} \quad (3.5)$$

$$\begin{aligned} [D_\varphi, \cos \varphi] &= -\sin \varphi - 2a \sum_{i=0}^{k-1} \sin \left(\varphi + \frac{i\pi}{k} \right) \cos \frac{i\pi}{k} \mathcal{R}^{k+2i} \mathcal{I} \\ &\quad + 2b \sum_{i=0}^{k-1} \cos \left(\varphi + \frac{i\pi}{k} \right) \sin \frac{i\pi}{k} \mathcal{R}^{2i} \mathcal{I}, \end{aligned} \quad (3.6)$$

$$\begin{aligned} [D_\varphi, \sin \varphi] &= \cos \varphi + 2a \sum_{i=0}^{k-1} \sin \left(\varphi + \frac{i\pi}{k} \right) \sin \frac{i\pi}{k} \mathcal{R}^{k+2i} \mathcal{I} \\ &\quad + 2b \sum_{i=0}^{k-1} \cos \left(\varphi + \frac{i\pi}{k} \right) \cos \frac{i\pi}{k} \mathcal{R}^{2i} \mathcal{I}, \end{aligned} \quad (3.7)$$

which can be easily established from (2.1) and (2.2).

It turns out, however, that the two sets of modified boson operators (3.1) and (3.2) do not have simple transformation properties under D_{2k} whenever $k > 1$. To cope with this drawback, it is advantageous to replace A, A^\dagger by k dependent pairs of modified boson operators $A_i, A_i^\dagger, i = 0, 1, \dots, k - 1$, defined by

$$\begin{aligned} A_i &= \frac{1}{\sqrt{2\omega}} \left[\cos\left(\varphi + \frac{i\pi}{k}\right) (\omega r + D_r) - \frac{1}{r} \sin\left(\varphi + \frac{i\pi}{k}\right) D_\varphi \right], \\ A_i^\dagger &= \frac{1}{\sqrt{2\omega}} \left[\cos\left(\varphi + \frac{i\pi}{k}\right) (\omega r - D_r) + \frac{1}{r} \sin\left(\varphi + \frac{i\pi}{k}\right) D_\varphi \right] \end{aligned} \quad (3.8)$$

and such that $A_0 = A$ and $A_0^\dagger = A^\dagger$. In the same way, B, B^\dagger may be replaced by $B_i, B_i^\dagger, i = 0, 1, \dots, k - 1$, obtained from (3.2) by substituting $\varphi + \frac{i\pi}{k}$ for φ (hence $B_0 = B, B_0^\dagger = B^\dagger$). For most purposes, though, it will be enough to only consider A_i, A_i^\dagger , since B_i, B_i^\dagger can be expressed in terms of them as²

$$B_i = \frac{2}{k} \sum_{j=0}^{k-1} A_j \sin \frac{(i-j)\pi}{k}, \quad B_i^\dagger = \frac{2}{k} \sum_{j=0}^{k-1} A_j^\dagger \sin \frac{(i-j)\pi}{k}. \quad (3.9)$$

Under D_{2k} , the operators $A_i, i = 0, 1, \dots, k - 1$, transform among themselves (and similarly for their Hermitian conjugates). We indeed obtain

$$\begin{aligned} \mathcal{R}A_i\mathcal{R}^{-1} &= A_{i+1}, & i = 0, 1, \dots, k - 2, & \quad \mathcal{R}A_{k-1}\mathcal{R}^{-1} = -A_0, \\ \mathcal{I}A_0\mathcal{I}^{-1} &= A_0, & \mathcal{I}A_i\mathcal{I}^{-1} &= -A_{k-i}, & i = 1, 2, \dots, k - 1. \end{aligned} \quad (3.10)$$

These relations can actually be simplified by extending the definition of A_i (and A_i^\dagger) to all values $i \in \mathbb{Z}$:

$$A_{\lambda k+i} = (-1)^\lambda A_i, \quad i = 0, 1, \dots, k - 1, \quad \lambda \in \mathbb{Z}. \quad (3.11)$$

We then get

$$\mathcal{R}A_i\mathcal{R}^{-1} = A_{i+1}, \quad \mathcal{I}A_i\mathcal{I}^{-1} = -A_{k-i}, \quad i \in \mathbb{Z}. \quad (3.12)$$

Among the k operators $A_i, i = 0, 1, \dots, k - 1$, there are only two linearly independent ones (e.g. $A_0 = A$ and $B_0 = B$). The A_i 's therefore satisfy $k - 2$ independent linear relations. We quote here some of them:

$$\sum_{i=0}^{k-1} (-1)^i A_i = 0, \quad \sum_{i=0}^{k-1} A_i \cos \frac{i\pi}{k} = \frac{k}{2} A_0, \quad (3.13)$$

$$A_i \cos \frac{i\pi}{k} = \frac{1}{2} (A_0 + A_{2i}), \quad B_i \sin \frac{i\pi}{k} = \frac{1}{2} (A_0 - A_{2i}), \quad i \in \mathbb{Z}, \quad (3.14)$$

which are easily derived through elementary trigonometry and will subsequently play an important role.

The commutation relations of $A_i, A_i^\dagger, i = 0, 1, \dots, k - 1$, can be determined from equations (2.4) and (3.3), as well as from the relations obtained from (3.4) to (3.7) by acting with \mathcal{R}^i on both sides. After a rather lengthy but straightforward calculation, we get the simple results

$$[A_i, A_j] = [A_i^\dagger, A_j^\dagger] = 0, \quad (3.15)$$

² Up to equation (3.8), all mentioned results are valid for any odd k value, including $k = 1$. This is not the case for equation (3.9), nor for some formulas to be given in the following, which require $k \geq 3$. From now on, we therefore restrict ourselves to such values.

$$\begin{aligned}
 [A_i, A_j^\dagger] &= [A_j, A_i^\dagger] = \cos \frac{(j-i)\pi}{k} + 2a \sum_l \cos \frac{(l-i)\pi}{k} \cos \frac{(l-j)\pi}{k} \mathcal{R}^{k+2l} \mathcal{I} \\
 &\quad + 2b \sum_l \sin \frac{(l-i)\pi}{k} \sin \frac{(l-j)\pi}{k} \mathcal{R}^{2l} \mathcal{I}
 \end{aligned} \tag{3.16}$$

for $i, j = 0, 1, \dots, k-1$. Note that from now on, all summations over l run from 0 to $k-1$.

4. Modified boson oscillator Hamiltonians and superintegrability of H_k for odd $k \geq 3$

The modified boson operators $A_i, A_i^\dagger, i = 0, 1, \dots, k-1$, can be used to define some modified oscillator Hamiltonians as

$$H_i = \frac{1}{2} \{A_i^\dagger, A_i\}, \quad i = 0, 1, \dots, k-1.$$

From (3.10), it follows that

$$\begin{aligned}
 \mathcal{R}H_i\mathcal{R}^{-1} &= H_{i+1}, & i = 0, 1, \dots, k-2, & \quad \mathcal{R}H_{k-1}\mathcal{R}^{-1} = H_0, \\
 \mathcal{I}H_0\mathcal{I}^{-1} &= H_0, & \mathcal{I}H_i\mathcal{I}^{-1} &= H_{k-i}, & i = 1, 2, \dots, k-1,
 \end{aligned} \tag{4.1}$$

so that the H_i 's transform among themselves under D_{2k} .

The explicit expression of H_i in terms of r, φ, D_r and D_φ can be easily found by first considering $i = 0$ and then acting with \mathcal{R}^i on the result. It is given by

$$\begin{aligned}
 2\omega H_i &= -\cos^2\left(\varphi + \frac{i\pi}{k}\right) D_r^2 + \frac{1}{r} \sin\left(\varphi + \frac{i\pi}{k}\right) \cos\left(\varphi + \frac{i\pi}{k}\right) (D_r D_\varphi + D_\varphi D_r) \\
 &\quad - \frac{1}{r^2} \sin^2\left(\varphi + \frac{i\pi}{k}\right) D_\varphi^2 \\
 &\quad - \frac{1}{r} \left[\sin^2\left(\varphi + \frac{i\pi}{k}\right) + 2a \sum_l \cos^2 \frac{(l-i)\pi}{k} \mathcal{R}^{k+2l} \mathcal{I} + 2b \sum_l \sin^2 \frac{(l-i)\pi}{k} \mathcal{R}^{2l} \mathcal{I} \right] D_r \\
 &\quad + \frac{1}{r^2} \left[-2 \sin\left(\varphi + \frac{i\pi}{k}\right) \cos\left(\varphi + \frac{i\pi}{k}\right) - 2a \sum_l \sin \frac{(l-i)\pi}{k} \cos \frac{(l-i)\pi}{k} \mathcal{R}^{k+2l} \mathcal{I} \right. \\
 &\quad \left. + 2b \sum_l \sin \frac{(l-i)\pi}{k} \cos \frac{(l-i)\pi}{k} \mathcal{R}^{2l} \mathcal{I} \right] D_\varphi + \omega^2 r^2 \cos^2\left(\varphi + \frac{i\pi}{k}\right).
 \end{aligned}$$

Elementary trigonometry now shows that the D_{2k} -extended Hamiltonian \mathcal{H}_k of equation (2.5) is directly connected with the H_i 's through the relation

$$2\omega \sum_{i=0}^{k-1} H_i = \frac{k}{2} \mathcal{H}_k. \tag{4.2}$$

Hence, it may be regarded as a modified boson oscillator Hamiltonian.

Next we may ask ourselves: what are the commutation relations of the H_i 's? As explained below, for our purpose, it is actually sufficient to determine the commutator of H_0 with $H_i, i = 1, 2, \dots, k-1$. On using the convention (3.11) for simplicity's sake, as well as equations (3.12), (3.15) and (3.16), we get after some calculations

$$\begin{aligned}
 [H_0, H_i] &= (A_0^\dagger A_i - A_i^\dagger A_0) \cos \frac{i\pi}{k} + 2a \sum_l \left[-(A_0^\dagger A_{2k+2l-i} + A_i^\dagger A_{k+2l}) \cos \frac{l\pi}{k} \cos \frac{(l-i)\pi}{k} \right. \\
 &\quad \left. + \frac{1}{2} (A_0^\dagger A_0 - A_{k+2l}^\dagger A_{k+2l}) \cos^2 \frac{(l-i)\pi}{k} \right]
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{2} \left(A_{2k+2l-i}^\dagger A_{2k+2l-i} - A_i^\dagger A_i \right) \cos^2 \frac{l\pi}{k} \Big] \mathcal{R}^{k+2l} \mathcal{I} \\
 & + 2b \sum_l \left[- \left(A_0^\dagger A_{k+2l-i} + A_i^\dagger A_{2l} \right) \sin \frac{l\pi}{k} \sin \frac{(l-i)\pi}{k} \right. \\
 & + \frac{1}{2} \left(A_0^\dagger A_0 - A_{2l}^\dagger A_{2l} \right) \sin^2 \frac{(l-i)\pi}{k} \\
 & + \left. \frac{1}{2} \left(A_{k+2l-i}^\dagger A_{k+2l-i} - A_i^\dagger A_i \right) \sin^2 \frac{l\pi}{k} \right] \mathcal{R}^{2l} \mathcal{I} \\
 & - \frac{k}{4} \sin \frac{2i\pi}{k} \sum_l \sin \frac{2l\pi}{k} (a^2 - 2ab\mathcal{R}^k + b^2) \mathcal{R}^{2l}. \tag{4.3}
 \end{aligned}$$

It may be observed that this relation is also valid for $i = 0$, leading to $[H_0, H_0] = 0$ as it should be.

On summing both sides of equation (4.3) over i from 0 to $k - 1$ and employing equations (3.13), (3.14), as well as some relations easily obtained from them, we arrive at the result $[H_0, \mathcal{H}_k] = 0$, where we have taken equation (4.2) into account. It only remains to apply \mathcal{R}^i and to use the invariance of \mathcal{H}_k under D_{2k} , together with equation (4.1), to infer that

$$[H_i, \mathcal{H}_k] = 0, \quad i = 0, 1, \dots, k - 1,$$

showing that $H_i, i = 0, 1, \dots, k - 1$, are integrals of motion of \mathcal{H}_k .

From them, we can form two D_{2k} invariants, namely their sum proportional to \mathcal{H}_k and their symmetrized product

$$\mathcal{Y}_{2k} = (2\omega)^k \sum_p H_{p(0)} H_{p(1)} \cdots H_{p(k-1)}, \tag{4.4}$$

where the summation runs over all $k!$ permutations of $0, 1, \dots, k - 1$. Projection in the D_{2k} identity representation space then leads to H_k , on one hand, and to an integral of motion Y_{2k} of the latter, on the other hand. Since Y_{2k} is clearly a differential operator of order $2k$, the conjecture of [1] will be proved for any odd $k \geq 3$ provided X_k and Y_{2k} are functionally independent.

This last point can be demonstrated by showing that $[\mathcal{X}_k, \mathcal{Y}_{2k}] \neq 0$ and that such a property remains true after projection. As $[\mathcal{X}_k, \mathcal{Y}_{2k}]$ is a $(2k + 1)$ th-order operator, it is enough to prove that the coefficient of one of the highest-order terms, namely $D_\varphi D_r^{2k}$, does not vanish. Such a term can be obtained by multiplying the term proportional to $D_\varphi D_r^2$ in one of the commutators $[\mathcal{X}_k, 2\omega H_i]$ by the terms proportional to D_r^2 in the remaining $2\omega H_j$'s ($j \neq i$) and summing over i , as well as over all permutations p .

On starting from the relation

$$\begin{aligned}
 [\mathcal{X}_k, 2\omega H_0] &= -[D_\varphi^2, 2\omega H_0] \\
 &= [r^2, 2\omega H_0] \mathcal{H}_k + [(r D_r)^2, 2\omega H_0] + 2 \left[(a\mathcal{R} + b) \sum_l \mathcal{R}^{2l} \mathcal{I}, 2\omega H_0 \right] r D_r \\
 &\quad + 2(a\mathcal{R} + b) \sum_l \mathcal{R}^{2l} \mathcal{I} [r D_r, 2\omega H_0] - \omega^2 [r^4, 2\omega H_0],
 \end{aligned}$$

it can be easily shown that the coefficient of $D_\varphi D_r^2$ in $[\mathcal{X}_k, 2\omega H_0]$ is equal to $-4 \sin \varphi \cos \varphi$. It is then enough to apply \mathcal{R}^i to this result to prove that the term $-4 \sin \left(\varphi + \frac{i\pi}{k} \right) \cos \left(\varphi + \frac{i\pi}{k} \right) D_\varphi D_r^2$ appears in $[\mathcal{X}_k, 2\omega H_i]$. Finally, we get the term $A D_\varphi D_r^{2k}$ with $A = -2^{-2k+4} k! k \sin k\varphi \cos k\varphi \neq 0$ in $[\mathcal{X}_k, \mathcal{Y}_{2k}]$, thereby showing that $[X_k, Y_{2k}] \neq 0$.

Taking into account that the truth of the conjecture of [1] is already known for $k = 1$ [3], we conclude that it is valid for any odd k .

5. The $k = 3$ case

To illustrate the approach followed in the present paper, it is worth considering the smallest value allowed for k , namely $k = 3$. In such a case, another proof of the superintegrability of H_k is indeed available through the use of other types of differential–difference operators, modified boson ones and modified boson oscillator Hamiltonians [13]. Here we plan to provide some links between those operators and the corresponding ones constructed in sections 2, 3 and 4.

The Hamiltonian H_3 appears when considering the three-body Hamiltonian of the CMW problem [6] and eliminating the centre-of-mass motion. The latter Hamiltonian

$$H_{\text{CMW}} = \sum_{i=1}^3 (-\partial_i^2 + \omega^2 x_i^2) + 2\alpha \left(\frac{1}{x_{12}^2} + \frac{1}{x_{23}^2} + \frac{1}{x_{31}^2} \right) + 6\beta \left(\frac{1}{y_{12}^2} + \frac{1}{y_{23}^2} + \frac{1}{y_{31}^2} \right),$$

where x_i , $i = 1, 2, 3$, denote the particle coordinates, $x_{ij} = x_i - x_j$, $i \neq j$, and $y_{ij} = x_i + x_j - 2x_k$, $i \neq j \neq k \neq i$, can indeed be separated into a centre-of-mass Hamiltonian $H_{\text{cm}} = -\partial_X^2 + \omega^2 X^2$ and a relative one H_{rel} , coinciding with H_3 , by defining $X = (x_1 + x_2 + x_3)/\sqrt{3}$, $r \cos \varphi = x_{12}/\sqrt{2}$ and $r \sin \varphi = y_{12}/\sqrt{6}$.

In [13], H_{CMW} has been transformed into a D_6 -extended Hamiltonian \mathcal{H}_{CMW} by realizing the 12 operators of D_6 in terms of the particle permutation operators K_{ij} and the inversion operator I_r in relative coordinate space and by introducing three differential–difference operators D_i , $i = 1, 2, 3$, defined as in equation (2.5) of [15]. Three sets of modified boson creation and annihilation operators $a_i^\dagger = (\omega x_i - D_i)/\sqrt{2\omega}$, $a_i = (\omega x_i + D_i)/\sqrt{2\omega}$, $i = 1, 2, 3$, have then been considered and used to build three modified oscillator Hamiltonians $h_i = \{a_i^\dagger, a_i\}/2$, $i = 1, 2, 3$. In terms of the latter,

$$\mathcal{H}_{\text{CMW}} = 2\omega \sum_i h_i, \tag{5.1}$$

while the sixth-order integral of motion of H_{CMW} has been derived from $(2\omega)^3 \sum_i h_i^3$ by projection in the D_6 identity representation space.

The correspondence $K_{ij} \leftrightarrow \mathcal{R}^{2k+3}\mathcal{I}$ and $K_{ij}I_r \leftrightarrow \mathcal{R}^{2k}\mathcal{I}$, where $(ijk) = (123)$, leads to equation (2.6) of [15], connecting the differential–difference operators of equation (2.2) with the D_i 's. As a result, we can express the three dependent modified annihilation operators A_i , $i = 0, 1, 2$, defined in (3.8), in terms of the three independent operators a_i , $i = 1, 2, 3$, as $A_0 = (a_1 - a_2)/\sqrt{2}$, $A_1 = -(a_2 - a_3)/\sqrt{2}$, $A_2 = (a_3 - a_1)/\sqrt{2}$. The alternative set of operators B_i , $i = 0, 1, 2$, is given by $B_0 = (a_1 + a_2 - 2a_3)/\sqrt{6}$, $B_1 = -(a_2 + a_3 - 2a_1)/\sqrt{6}$, $B_2 = (a_3 + a_1 - 2a_2)/\sqrt{6}$. From this, it can be easily checked that there is a single relation $A_0 - A_1 + A_2 = 0$ among the A_i 's (and similarly for the B_i 's) and that all relations in (3.13) and (3.14) reduce to that one for $k = 3$. Furthermore, the commutation relations (3.15) and (3.16) entirely agree with those that can be derived from the above-mentioned relations between the A_i 's and the a_i 's and from the commutation relations of the latter given in equation (4.2) of [13].

As we consider the modified boson oscillator Hamiltonians, we obtain the equations

$$\begin{aligned} H_0 &= \frac{1}{2} [h_1 + h_2 - \frac{1}{2} \{a_1^\dagger, a_2\} - \frac{1}{2} \{a_2^\dagger, a_1\}], \\ H_1 &= \frac{1}{2} [h_2 + h_3 - \frac{1}{2} \{a_2^\dagger, a_3\} - \frac{1}{2} \{a_3^\dagger, a_2\}], \\ H_2 &= \frac{1}{2} [h_3 + h_1 - \frac{1}{2} \{a_3^\dagger, a_1\} - \frac{1}{2} \{a_1^\dagger, a_3\}], \end{aligned} \tag{5.2}$$

where the extra anticommutators on the right-hand sides serve to compensate for the centre-of-mass degrees of freedom present in h_1, h_2, h_3 , but absent from H_0, H_1, H_2 . From equations (4.2), (5.1) and (5.2), it follows that

$$\mathcal{H}_3 = \frac{2}{3}\mathcal{H}_{\text{CMW}} - \frac{1}{3}\omega \sum_{i \neq j} \{a_i^\dagger, a_j\},$$

in agreement with the decomposition

$$\mathcal{H}_{\text{CMW}} = H_{\text{cm}} + \mathcal{H}_3, \quad H_{\text{cm}} = \frac{1}{3}\omega \{a_1^\dagger + a_2^\dagger + a_3^\dagger, a_1 + a_2 + a_3\} = -\partial_X^2 + \omega^2 X^2.$$

Finally, the operator \mathcal{Y}_6 , giving rise by projection to the sixth-order integral of motion of H_3 , could be expressed in terms of $a_i^\dagger, a_i, i = 1, 2, 3$, through equations (4.4) and (5.2).

6. Conclusion

In the present communication, we have provided an alternative approach to the construction of the sixth-order integral of motion of the CMW Hamiltonian, wherein the centre-of-mass motion has been eliminated from the very beginning, in contrast with the previous one based on the use of differential–difference operators related to the three particle coordinates on the line [13]. The present construction instead employs differential–difference operators D_r, D_φ on the relative-coordinate space. A key step in the construction has been the introduction of three dependent sets of modified boson creation and annihilation operators with simple exchange properties under the dihedral group D_6 connected with the CMW problem.

An analogous approach, taking advantage of some recent progress made in [15], has allowed us to prove the superintegrability of all TTW Hamiltonians H_k with odd $k \geq 3$. In such a case, the associated group being D_{2k} , we have been led to consider k dependent sets of modified boson creation and annihilation operators and the k modified oscillator Hamiltonians that can be built from them. In terms of the latter, the construction of the missing integral of motion Y_{2k} has proved easy. Furthermore, Y_{2k} has been shown to be functionally independent of the first integral of motion X_k , thereby completing the demonstration.

Since for even k values, the operator D_φ assumes a different form from that considered here for odd k , it is not yet clear whether the present formalism can be extended to such values and lead to a proof of the remaining part of the conjecture. This is left for a future study.

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References

- [1] Tremblay F, Turbiner A V and Winternitz P 2009 *J. Phys. A: Math. Theor.* **42** 242001
- [2] Friš J, Mandrosov V, Smorodinsky Ya A, Uhlir M and Winternitz P 1965 *Phys. Lett.* **16** 354
- [3] Winternitz P, Smorodinsky Ya A, Uhlir M and Friš J 1967 *Sov. J. Nucl. Phys.* **4** 444
- [4] Olshanetsky M A and Perelomov A M 1983 *Phys. Rep.* **94** 313
- [5] Calogero F and Marchioro C 1974 *J. Math. Phys.* **15** 1425
- [6] Wolfes J 1974 *J. Math. Phys.* **15** 1420
- [7] Tremblay F, Turbiner A V and Winternitz P 2010 *J. Phys. A: Math. Theor.* **43** 015202
- [8] Kalnins E G, Miller W Jr and Pogosyan G S 2009 Superintegrability and higher order constants for classical and quantum systems arXiv:0912.2278
- [9] Kalnins E G, Kress J M and Miller W Jr 2009 Families of classical subgroup separable superintegrable systems arXiv:0912.3158

- [10] Dunkl C F 1989 *Trans. Am. Math. Soc.* **311** 167
- [11] Polychronakos A P 1992 *Phys. Rev. Lett.* **69** 703
- [12] Brink L, Hansson T H and Vasiliev M A 1992 *Phys. Lett. B* **286** 109
- [13] Quesne C 1995 *Mod. Phys. Lett. A* **10** 1323
- [14] Khare A and Quesne C 1998 *Phys. Lett. A* **250** 33
- [15] Quesne C 2010 Exchange operator formalism for an infinite family of solvable and integrable quantum systems on a plane *Mod. Phys. Lett. A* **25** 15 (arXiv:[0910.2151](https://arxiv.org/abs/0910.2151))